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The asymptotic stress field for a rigid circular inclusion at the interface of two bonded dissimilar elastic half-space materials

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Abstract

The paper considers three-dimensional interface inclusion problems. The axisymmetric elastostatics problem of a rigid circular inclusion at the interface between two perfectly bonded dissimilar elastic half spaces is analyzed. Based on the representations of displacements and stresses in terms of Love's strain potential and the Hankel transform technique, the mixed boundary value problem associated with a rigid circular inclusion at the interface reduces to a pair of simultaneous integral equations for the stress jumps across the inclusion, which are further transformed to a single singular integral equation. For the case of uniform axial and radial tensions at infinity, the asymptotic stresses near the inclusion front are obtained and they exhibit the oscillatory singularity. Meanwhile, the magnitude of the singularity for the interface inclusion depends on the material constants of the upper and lower half spaces, the dependence of singularity coefficients on material constants for interface inclusion problems, however, is different from that for interface crack problems. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

The performance of materials is altered by the presence of inhomogeneities such as cracks, cavities, and inclusions, etc. (Mura, 1982). As well known, cracks contained in a medium usually weak the mechanical reliability of materials. For a rigid flat inclusion embedded in a medium, it can also result in the concentration of stresses and further affect the behavior of materials. From the viewpoint of inhomogeneities in solids, cracks and rigid inclusions are the two extreme cases of an inhomogeneity, namely, $\mu \to 0$ for a crack, and $\mu \to \infty$ for a rigid inclusion, where μ is the shear modulus of the inhomogeneity phase. Therefore, the problems involving inhomogeneities including cracks and inclusions in an elastic material

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have become the subject of extensive investigations in recent years due to the importance of the mechanical reliability of materials in critical design. So the study of elastic field disturbed by a rigid inclusion is as important as that by a crack in medium. A review given by Mura (1988) contains a large amount of literature dealing with inclusion problems.

On the other hand, with the wide application of composite materials, the problems of a bimaterial interface have received much attention in recent years. Considerable researches focus on interface crack problems including two-dimensional and three-dimensional problems. In comparison with interface crack problems, the study of interface inclusion problems, however, is rather limited, and most investigations in this area is mainly concentrated on two-dimensional problems, i.e. a rigid line inclusion is placed at the interface of a bimaterial consisting of two half planes of dissimilar elastic media. For example, the characteristics of stress field for rigid line inclusions at the interface of two perfectly bonded dissimilar semi-infinite elastic media has been investigated by some researchers (see e.g. Dundurs and Markenscoff, 1989; Markenscoff et al., 1994; Markenscoff and Ni, 1996; Ballarini, 1990; Wu, 1990; Jiang and Liu, 1992; Asundi and Deng, 1995; Boniface and Hasebe, 1998; etc.). Ballarini (1990) considered the problem of a rigid line inclusion at the interface of a bimaterial subjected to constant loads at infinity, and derived the analytical solution and stress singularity coefficients at the rigid inclusion tip using the method of complex potential. A similar problem to the above has also been studied by Jiang and Liu (1992) solving the Riemann–Hilbert problem derived from a rigid line inclusion at a bimaterial interface. For the cases of uniform biaxial tension and of inclusion loaded by concentrated forces or moment, Dundurs and Markenscoff (1989) and Markenscoff et al. (1994) analyzed the problems of interface anticracks, or rigid inclusions, of a bimaterial and obtained the Green's functions in explicit form by means of solving the system of governing coupled integral equations. Recently, two-dimensional rigid interface line inclusion problems have been further investigated by Markenscoff and Ni (1996). The two-dimensional elasticity problem of a rigid elliptic inclusion, instead of a rigid line inclusion, at the interface between two dissimilar isotropic elastic half planes has been presented and solved by Boniface and Hasebe (1998) using the method of Muskhelishvili's complex potential and the conformal mapping technique. Work on anisotropic rigid interface line inclusion can be found in Wu (1990) and Asundi and Deng (1995).

In comparison with the two-dimensional interface inclusion problems, three-dimensional interface inclusions are more practical and more important from an engineering viewpoint. However, to the best of the author's knowledge, few solutions are available for three rigid interface inclusion problems, although a number of solutions for three-dimensional problems in a homogeneous elastic material rather than interface inclusion problems and interface crack problems (see e.g. Qu and Xue, 1998; Rao and Hasebe, 1995; Nakamura, 1991; Saxena and Dhaliwal, 1990; Kassir and Bregman, 1972; Wills, 1972; Erdogan, 1965; etc.) have been reported. Selvadurai (1985) developed a set of bounds which can be used to estimate the asymptotic rotational stiffness of a rigid elliptical disc inclusion at the interface of a bimaterial.

This paper considers three-dimensional rigid interface inclusion problems. The axisymmetric elastostatics problem of a rigid circular inclusion at the interface between two perfectly bonded dissimilar isotropic elastic half spaces is analyzed. This paper is organized as follows: The basic theory for solving a class of axisymmetric elastostatics problem is outlined in Section 2. In Section 3, superposition is utilized to separate the problem under consideration into two auxiliary problems: one relating to a uniform elastic field and the other corresponding to a singular elastic field. In Section 4, by using the Hankel transform, we reduce the mixed boundary value problem involving a rigid circular interface inclusion to a pair of simultaneous integral equations for the stress jumps across the inclusion, which are further rewritten as a single singular integral equation. In Section 5, for the case of uniform axial and radial tensions at infinity, the asymptotic stress field near the inclusion front at the inclusion surfaces is obtained. The results indicate

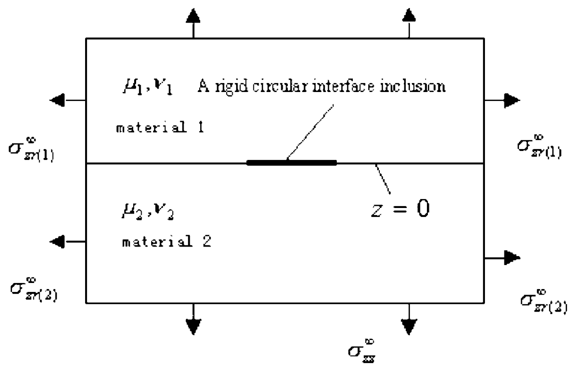

Fig. 1. A rigid circular inclusion at the interface of a bimaterial.

that the stresses exhibit the oscillatory singularity similar to that for an interface crack. In Section 6, two particular cases are considered, and conclusion are made.

2. Basic theory

Consider a class of axisymmetric elastostatics problems associated with a rigid circular inclusion at the interface between two perfectly bonded dissimilar elastic half spaces, as shown in Fig. 1. In order to solve axisymmetric elastostatics problems, it is convenient to adopt a cylindrical coordinate system (r, θ, z) with the origin at the center of the inclusion. Let the inclusion be situated at the region $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant 2\pi$ and $z = 0$, and medium I with moduli μ_1, ν_1 and medium II with moduli μ_2, ν_2 occupy the upper and lower half spaces, S_1 and S_2, where μ_i, ν_i $(i = 1, 2)$ denote the shear modulus and Poisson's ratio, respectively.

Based on a formulation from Love's strain potential approach, the displacements in a bimaterial free of body forces may be expressed in terms of the derivatives of $\varphi_{(1)}(r, z)$ and $\varphi_{(2)}(r, z)$, which correspond to the potentials for the upper and lower half spaces, respectively, viz.

$$u_{r(i)}(r, z) = -\frac{\partial^2 \varphi_{(i)}}{\partial r \partial z}, \tag{1}$$

$$u_{z(i)}(r, z) = 2(1 - \nu_i)\nabla^2 \varphi_{(i)} - \frac{\partial^2 \varphi_{(i)}}{\partial z^2}, \tag{2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{3}$$

is the Laplacian operator referred to a cylindrical coordinate system (r, θ, z), the subscript (i), $i = 1, 2$, denotes "in region S_i", and the potentials $\varphi_{(i)}$ are governed by the bi-harmonic equation

$$\nabla^2 \nabla^2 \varphi_{(i)} = 0. \tag{4}$$

Under such circumstances, the stresses may be given in terms of $\varphi_{(i)}$ as follows:

$$\sigma_{rr(i)}(r,z) = 2\mu_i \frac{\partial}{\partial z}\left[v_i \nabla^2 \varphi_{(i)} - \frac{\partial^2 \varphi_{(i)}}{\partial r^2}\right], \tag{5}$$

$$\sigma_{\theta\theta(i)}(r,z) = 2\mu_i \frac{\partial}{\partial z}\left[v_i \nabla^2 \varphi_{(i)} - \frac{1}{r}\frac{\partial \varphi_{(i)}}{\partial r}\right], \tag{6}$$

$$\sigma_{zz(i)}(r,z) = 2\mu_i \frac{\partial}{\partial z}\left[(2 - v_i)\nabla^2 \varphi_{(i)} - \frac{\partial^2 \varphi_{(i)}}{\partial z^2}\right], \tag{7}$$

$$\sigma_{zr(i)}(r,z) = 2\mu_i \frac{\partial}{\partial r}\left[(1 - v_i)\nabla^2 \varphi_{(i)} - \frac{\partial^2 \varphi_{(i)}}{\partial z^2}\right]. \tag{8}$$

3. Statement of the problem

In this section, we are concerned with the problem of a rigid circular inclusion at the interface of a bimaterial subjected to constant stresses along the z-axis (axial) direction and along the radial direction at infinity. Hence the boundary conditions at infinity can stated below

$$\sigma_{zz}(r,z) = \sigma_{zz}^{\infty}, \quad \sigma_{rr(1)}(r,z) = \sigma_{rr(1)}^{\infty}, \quad \sigma_{rr(2)}(r,z) = \sigma_{rr(2)}^{\infty}, \qquad \sqrt{r^2+z^2} \to \infty. \tag{9}$$

Clearly, it follows from the linearity of the problem that the problem under consideration can be solved by superposition. In other words, each of the field variables can be separated into two parts: (a) the homogeneous elastic field for a bimaterial with no inclusion subjected to the same loads as expressed by Eq. (9) at infinity, and (b) the singular elastic field for a bimaterial with an interface inclusion for which the elastic displacements along the inclusion are prescribed as the negative of those produced by the former. Therefore, one can write the potential $\varphi_{(i)}$ as

$$\varphi_{(i)} = \varphi_{(i)}^{\mathrm{H}} + \varphi_{(i)}^{\mathrm{S}}, \quad i = 1,2, \tag{10}$$

where a variable with the superscript H or S denotes the corresponding one in the homogeneous or singular elastic fields, respectively.

For the problem (a), using the continuity conditions along the bimaterial interface:

$$u_{z(1)}^{\mathrm{H}}(r,0^+) = u_{z(2)}^{\mathrm{H}}(r,0^-), \quad r \geqslant 0, \tag{11}$$

$$u_{r(1)}^{\mathrm{H}}(r,0^+) = u_{r(2)}^{\mathrm{H}}(r,0^-), \quad r \geqslant 0, \tag{12}$$

$$\sigma_{zz(1)}^{\mathrm{H}}(r,0^+) = \sigma_{zz(2)}^{\mathrm{H}}(r,0^-), \quad r \geqslant 0, \tag{13}$$

$$\sigma_{zr(1)}^{\mathrm{H}}(r,0^+) = \sigma_{zr(2)}^{\mathrm{H}}(r,0^-), \quad r \geqslant 0, \tag{14}$$

where the superscript $+(-)$ represents the limit as the interface is approached from the upper(lower) half-space $S_1(S_2)$, we easily get that the potentials in this case are given by

$$\varphi_{(i)}^{\mathrm{H}} = \frac{1}{12\mu_i(1+v_i)}\{[2(2-v_i)\sigma_{rr(i)}^{\infty} + (1-2v_i)\sigma_{zz}^{\infty}]z^3 - 3[(1-v_i)\sigma_{rr(i)}^{\infty} - v_i\sigma_{zz}^{\infty}]zr^2\} \tag{15}$$

and that the far-field stresses obey the following relation

$$\sigma^{\infty}_{rr(2)} = \frac{\mu_2(1+\nu_2)(1-\nu_1)}{\mu_1(1+\nu_1)(1-\nu_2)}\sigma^{\infty}_{rr(1)} + \frac{\mu_1\nu_2 - \mu_2\nu_1 + \nu_1\nu_2(\mu_1-\mu_2)}{\mu_1(1+\nu_1)(1-\nu_2)}\sigma^{\infty}_{zz}. \tag{16}$$

As a result, substitution of the potentials (15) into Eqs. (1) and (2) yields the displacements at the interface

$$u^{\mathrm{H}}_{z(1)}(r,0^+) = u^{\mathrm{H}}_{z(2)}(r,0^-) = 0, \quad r \geqslant 0, \tag{17}$$

$$u^{\mathrm{H}}_{r(1)}(r,0^+) = u^{\mathrm{H}}_{r(2)}(r,0^-) = \frac{rb_r}{2}, \quad r \geqslant 0, \tag{18}$$

with

$$b_r = \frac{(1+\kappa_1)\sigma^{\infty}_{rr(1)} - (3-\kappa_1)\sigma^{\infty}_{zz}}{\mu_1(7-\kappa_1)}, \tag{19}$$

where $\kappa_1 = 3 - 4\nu_1$. In particular, if the upper and lower half spaces are of the same material, it means that a rigid circular inclusion is embedded in an infinite elastic solid. In this case, the radial displacement component from Eq. (18) reduces to

$$u^{\mathrm{H}}_{r}(r,0) = \frac{1}{2\mu(1+\nu)}[(1-\nu)\sigma^{\infty}_{rr} + \nu\sigma^{\infty}_{zz}]r, \quad r \geqslant 0. \tag{20}$$

For the problem (b), singular elastic field is produced by the presence of a rigid inclusion at the interface. In fact, of practical importance is this case, and the disturbed elastic field may be determined from the following mixed boundary conditions:

$$u^{\mathrm{S}}_{z(1)}(r,0^+) = u^{\mathrm{S}}_{z(2)}(r,0^-), \quad u^{\mathrm{S}}_{r(1)}(r,0^+) = u^{\mathrm{S}}_{r(2)}(r,0^-), \qquad r \geqslant 0, \tag{21}$$

$$\sigma^{\mathrm{S}}_{zz(1)}(r,0^+) = \sigma^{\mathrm{S}}_{zz(2)}(r,0^-), \quad \sigma^{\mathrm{S}}_{zr(1)}(r,0^+) = \sigma^{\mathrm{S}}_{zr(2)}(r,0^-), \qquad r > a, \tag{22}$$

$$u^{\mathrm{S}}_{z(1)}(r,0^+) = -f(r), \quad u^{\mathrm{S}}_{r(1)}(r,0^+) = -g(r), \qquad 0 \leqslant r \leqslant a, \tag{23}$$

where $f(r)$ and $g(r)$ are given by Eqs. (17) and (18), respectively, i.e.

$$f(r) = 0, \quad g(r) = \frac{rb_r}{2}. \tag{24}$$

As an unbounded region is under consideration, the regularity conditions at infinity

$$u^{\mathrm{S}}_{(i)}(r,z) \to 0, \quad \sigma^{\mathrm{S}}_{(i)}(r,z) \to 0, \qquad \sqrt{r^2+z^2} \to \infty \tag{25}$$

must be supplemented.

4. Governing equation

In order to solve the problem (b), it is convenient to adopt the Hankel transform technique. Following Sneddon (1951), define the mth order Hankel transform

$$\mathscr{H}_m[\varphi] = \int_0^{\infty} r\varphi(r)J_m(r\xi)\,\mathrm{d}r, \tag{26}$$

where J_m is the Bessel function of the first kind of order m. The zeroth order Hankel transform with respect to r of the bi-harmonic equation (4) leads to an ordinary differential equation for $\mathscr{H}_0[\varphi^{\mathrm{S}}_{(i)}](\xi,z)$. On account of the regularity conditions at infinity, unbounded terms are cast away and the general solutions are determined. Further, the inverse Hankel transform of the general solutions will give that the potentials can be expressed as the following integrals

$$\varphi^{\mathrm{S}}_{(1)}(r,z) = \int_0^\infty \xi^{-2}(A_1 + 2v_1B_1 + B_1z\xi)\mathrm{e}^{-z\xi}J_0(r\xi)\,\mathrm{d}\xi, \quad z \geqslant 0, \tag{27}$$

$$\varphi^{\mathrm{S}}_{(2)}(r,z) = \int_0^\infty \xi^{-2}(A_2 + 2v_2B_2 - B_2z\xi)\mathrm{e}^{z\xi}J_0(r\xi)\,\mathrm{d}\xi, \quad z \leqslant 0, \tag{28}$$

where $A_i = A_i(\xi)$ and $B_i = B_i(\xi)$, $i = 1, 2$, are new unknown functions in ξ to be determined from given boundary conditions (21)–(23). For convenience, in what follows the superscript S corresponding the singular elastic field is omitted and we restrict our attention to the problem (b). In fact, the difference between two elastic fields given by the potentials $\varphi^{\mathrm{S}}_{(i)}(r,z)$ and $\varphi_{(i)}(r,z)$ is a homogeneous elastic field, so it does not disturb the distribution of stresses. Substitution of the expression for $\varphi_{(1)}(r,z)$ given by Eq. (27) into Eqs. (1), (2), (7) and (8) yields the expressions for the displacements and stresses in the upper half space as follows:

$$u_{z(1)} = -\int_0^\infty [A_1 + 2(1-v_1)B_1 + B_1z\xi]\mathrm{e}^{-z\xi}J_0(r\xi)\,\mathrm{d}\xi, \quad z \geqslant 0, \tag{29}$$

$$u_{r(1)} = -\int_0^\infty [A_1 - (1-2v_1)B_1 + B_1z\xi]\mathrm{e}^{-z\xi}J_1(r\xi)\,\mathrm{d}\xi, \quad z \geqslant 0, \tag{30}$$

$$\sigma_{zz(1)} = 2\mu_1\int_0^\infty \xi(A_1 + B_1 + B_1z\xi)\mathrm{e}^{-z\xi}J_0(r\xi)\,\mathrm{d}\xi, \quad z \geqslant 0, \tag{31}$$

$$\sigma_{zr(1)} = 2\mu_1\int_0^\infty \xi(A_1 + B_1z\xi)\mathrm{e}^{-z\xi}J_1(r\xi)\,\mathrm{d}\xi, \quad z \geqslant 0. \tag{32}$$

Similarly, the expressions for the displacements and stresses in the lower half space are obtainable by inserting $\varphi_{(2)}(r,z)$ given by Eq. (28) into Eqs. (1), (2), (7) and (8). They are

$$u_{z(2)} = -\int_0^\infty [A_2 + 2(1-v_2)B_2 - B_2z\xi]\mathrm{e}^{z\xi}J_0(r\xi)\,\mathrm{d}\xi, \quad z \leqslant 0, \tag{33}$$

$$u_{r(2)} = \int_0^\infty [A_2 - (1-2v_2)B_2 - B_2z\xi]\mathrm{e}^{z\xi}J_1(r\xi)\,\mathrm{d}\xi, \quad z \leqslant 0, \tag{34}$$

$$\sigma_{zz(2)} = -2\mu_2\int_0^\infty \xi(A_2 + B_2 - B_2z\xi)\mathrm{e}^{z\xi}J_0(r\xi)\,\mathrm{d}\xi, \quad z \leqslant 0, \tag{35}$$

$$\sigma_{zr(2)} = 2\mu_2\int_0^\infty \xi(A_2 - B_2z\xi)\mathrm{e}^{z\xi}J_1(r\xi)\,\mathrm{d}\xi, \quad z \leqslant 0. \tag{36}$$

Substituting Eqs. (29), (30), (33) and (34) into Eq. (21) leads to a system of linear algebraic equations for A_i and B_i, and solving this system for A_2 and B_2 in terms of A_1 and B_1 yields

$$A_2 = \frac{-2A_1 + (\kappa_1\kappa_2 - 1)B_1}{2\kappa_2}, \qquad B_2 = \frac{2A_1 + B_1}{\kappa_2}, \tag{37}$$

where $\kappa_2 = 3 - 4v_2$.

To reduce the remaining mixed boundary conditions (22) and (23) to a system of integral equations, from Eqs. (29) and (30) for $z = 0$ we have

$$u_z(r) = -\int_0^\infty [A_1 + 2(1-\nu_1)B_1]J_0(r\xi)\,\mathrm{d}\xi, \quad r \geqslant 0, \tag{38}$$

$$u_r(r) = -\int_0^\infty [A_1 - (1-2\nu_1)B_1]J_1(r\xi)\,\mathrm{d}\xi, \quad r \geqslant 0, \tag{39}$$

where $u_z(r)$ and $u_r(r)$ denote the axial and radial components of displacement vector at the interface, $z = 0$, respectively. It is convenient for later use to introduce the stress jumps across the interface,

$$\Delta\sigma_{zz}(r) = \sigma_{zz(1)}(r, 0^+) - \sigma_{zz(2)}(r, 0^-), \quad r \geqslant 0, \tag{40}$$

$$\Delta\sigma_{zr}(r) = \sigma_{zr(1)}(r, 0^+) - \sigma_{zr(2)}(r, 0^-), \quad r \geqslant 0. \tag{41}$$

Then it immediately follows from the boundary conditions (22) that

$$\Delta\sigma_{zz}(r) = 0, \quad r > a, \tag{42}$$

$$\Delta\sigma_{zr}(r) = 0, \quad r > a. \tag{43}$$

Now, inserting Eq. (37) into the representations of stresses at the interface and subtracting Eqs. (35) and (36) from Eqs. (31) and (32) for $z = 0$, respectively, yield

$$\Delta\sigma_{zz}(r) = \int_0^\infty \left[\frac{1}{\gamma_2}(2A_1 + B_1) + \frac{\kappa_1}{\gamma_1}B_1\right]\xi J_0(r\xi)\,\mathrm{d}\xi, \quad r \geqslant 0, \tag{44}$$

$$\Delta\sigma_{zr}(r) = \int_0^\infty \left[\frac{1}{\gamma_2}(2A_1 + B_1) - \frac{\kappa_1}{\gamma_1}B_1\right]\xi J_1(r\xi)\,\mathrm{d}\xi, \quad r \geqslant 0, \tag{45}$$

where

$$\gamma_1 = \frac{\kappa_1}{\mu_1 + \kappa_1\mu_2}, \qquad \gamma_2 = \frac{\kappa_2}{\mu_2 + \kappa_2\mu_1}. \tag{46}$$

Taking the inverse Hankel transform of Eqs. (44) and (45) and solving the resulting linear equations for A_1 and B_1 in terms of the stress jumps $\Delta\sigma_{zz}(r)$ and $\Delta\sigma_{zr}(r)$, we find A_1 and B_1 to be

$$A_1 = \frac{1}{4}\left(\gamma_2 - \frac{\gamma_1}{\kappa_1}\right)\int_0^a \Delta\sigma_{zz}(r)rJ_0(r\xi)\,\mathrm{d}r + \frac{1}{4}\left(\gamma_2 - \frac{\gamma_1}{\kappa_1}\right)\int_0^a \Delta\sigma_{zr}(r)rJ_1(r\xi)\,\mathrm{d}r, \tag{47}$$

$$B_1 = \frac{\gamma_1}{2\kappa_1}\int_0^a \Delta\sigma_{zz}(r)rJ_0(r\xi)\,\mathrm{d}r - \frac{\gamma_1}{2\kappa_1}\int_0^a \Delta\sigma_{zr}(r)rJ_1(r\xi)\,\mathrm{d}r. \tag{48}$$

Next, we will eliminate A_1 and B_1 from Eqs. (38), (39) and Eqs. (47), (48). To this end, multiply Eq. (38) by $-r(x^2 - r^2)^{-(1/2)}$, then integrate with respect to r between the limits 0 and x, and change the order of integration at the right-hand side of Eq. (38). Using the known result (A.5) we get

$$-\int_0^x \frac{ru_z(r)}{\sqrt{x^2 - r^2}}\,\mathrm{d}r = \int_0^\infty [A_1 + 2(1-\nu_1)B_1]\frac{\sin(x\xi)}{\xi}\,\mathrm{d}\xi. \tag{49}$$

Differentiating Eq. (49) with respect to x, then substituting Eqs. (47) and (48) into Eq. (49), and using the known results (A.1) and (A.3), we obtain

$$-\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{ru_z(r)}{\sqrt{x^2-r^2}}\,\mathrm{d}r = \frac{\gamma_1+\gamma_2}{4}\int_x^a \frac{r\Delta\sigma_{zz}(r)}{\sqrt{r^2-x^2}}\,\mathrm{d}r + \frac{\gamma_1-\gamma_2}{4}\left[x\int_0^x \frac{\Delta\sigma_{zr}(r)}{\sqrt{x^2-r^2}}\,\mathrm{d}r - \int_0^a \Delta\sigma_{zr}(r)\,\mathrm{d}r\right], \quad x \geqslant 0. \tag{50}$$

Similarly, for the representation of $u_r(r)$ given by Eq. (39), multiplying Eq. (39) by r, then differentiating Eq. (39) with respect to r, and finally dividing Eq. (39) by $-r$, we find

$$-\frac{1}{r}[ru_r(r)]' = \int_0^\infty [A_1 - (1-2\nu_1)B_1]\xi J_0(r\xi)\,\mathrm{d}\xi, \tag{51}$$

where the prime denotes the differentiation with respect to r, and in deriving the above relation we have used Eq. (A.14).

Just as the derivation of the relationship (50), in view of Eqs. (A.5), (A.2) and (A.4) an entirely analogous procedure reduces Eq. (51) to the following relationship:

$$-\int_0^x \frac{[ru_r(r)]'}{\sqrt{x^2-r^2}}\,\mathrm{d}r = -\frac{\gamma_1-\gamma_2}{4}\int_0^x \frac{r\Delta\sigma_{zz}(r)}{\sqrt{x^2-r^2}}\,\mathrm{d}r + \frac{\gamma_1+\gamma_2}{4}x\int_x^a \frac{\Delta\sigma_{rz}(r)}{\sqrt{r^2-x^2}}\,\mathrm{d}r, \quad x \geqslant 0. \tag{52}$$

Therefore, we obtain the relationships between the displacements at the interface and the stress jumps across the interface, i.e. they are governed by Eqs. (50) and (52). It is well known the fact that for the axisymmetric problem of a rigid interface inclusion considered in this paper, the displacements $u_z(r)$ and $u_r(r)$ at the region $0 \leqslant r \leqslant a$ are prescribed, given by Eq. (23). Consequently, from Eqs. (50) and (52) we arrive at the pair of simultaneous governing integral equations for the stress jumps across the interface as follows:

$$\alpha\int_x^a \frac{r\Delta\sigma_{zz}(r)}{\sqrt{r^2-x^2}}\,\mathrm{d}r + \beta\left[x\int_0^x \frac{\Delta\sigma_{zr}(r)}{\sqrt{x^2-r^2}}\,\mathrm{d}r - \int_0^a \Delta\sigma_{zr}(r)\,\mathrm{d}r\right] = b_z(x), \quad 0 \leqslant x < a, \tag{53}$$

$$-\beta\int_0^x \frac{r\Delta\sigma_{zz}(r)}{\sqrt{x^2-r^2}}\,\mathrm{d}r + \alpha x\int_x^a \frac{\Delta\sigma_{zr}(r)}{\sqrt{r^2-x^2}}\,\mathrm{d}r = b_r(x), \quad 0 \leqslant x < a, \tag{54}$$

where

$$\alpha = \frac{\gamma_1+\gamma_2}{4}, \qquad \beta = \frac{\gamma_1-\gamma_2}{4} \tag{55}$$

and

$$b_z(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{ru_z(r)}{\sqrt{x^2-r^2}}\,\mathrm{d}r, \qquad b_r(x) = -\int_0^x \frac{[ru_r(r)]'}{\sqrt{x^2-r^2}}\,\mathrm{d}r. \tag{56}$$

Evidently, for the rigid inclusion problem under consideration, it is easily seen from Eqs. (23) and (24) that

$$b_z(x) = 0, \quad b_r(x) = b_r x. \tag{57}$$

Hence, once the stress jumps across the interface $\Delta\sigma_{zz}(r)$ and $\Delta\sigma_{zr}(r)$ are determined, unknown functions A_1 and B_1, and further A_2 and B_2, can be found from Eqs. (47) and (48), and Eq. (37), respectively.

To solve the pair of Eqs. (53) and (54), a further simplification is achieved by setting

$$\int_x^a \frac{r\Delta\sigma_{zz}(r)}{\sqrt{r^2-x^2}}\,\mathrm{d}r = \varphi(x), \quad 0 \leqslant x < a \tag{58}$$

and

$$\int_x^a \frac{\Delta\sigma_{zr}(r)}{\sqrt{r^2 - x^2}}\,dr = \eta(x), \quad 0 \leqslant x < a. \tag{59}$$

Under these circumstances, we multiply Eqs. (58) and (59) by $x(x^2 - \rho^2)^{-1/2}$, and integrate with respect to x between the limits ρ and a ($\rho < a$). Reversing the order of integration at the left-hand side and employing the equality (A.6) we can express the stress jumps across the interface $\Delta\sigma_{zz}(r)$ and $\Delta\sigma_{zr}(r)$ in terms of $\varphi(r)$ and $\eta(r)$, respectively, in the form

$$\Delta\sigma_{zz}(r) = -\frac{2}{\pi r}\frac{d}{dr}\int_r^a \frac{x\varphi(x)}{\sqrt{x^2 - r^2}}\,dx, \quad 0 \leqslant x < a, \tag{60}$$

$$\Delta\sigma_{zr}(r) = -\frac{2}{\pi}\frac{d}{dr}\int_r^a \frac{x\eta(x)}{\sqrt{x^2 - r^2}}\,dx, \quad 0 \leqslant x < a, \tag{61}$$

where in the last step in deriving the above results, we have replaced ρ with r for convenience.

Now, inserting Eq. (60) into the first integral at the left-hand side of Eq. (54), we find

$$\int_0^x \frac{r\Delta\sigma_{zz}(r)}{\sqrt{x^2 - r^2}}\,dr = \frac{1}{\pi}\int_0^a \varphi(s)\left[\frac{1}{s+x} - \frac{1}{s-x}\right]ds, \tag{62}$$

the derivation of which is given in Appendix B. Here, the integral involving the second term in the bracket is understood in the sense of the Cauchy principal value.

Similarly, from Eq. (61) we can deduce the following relationship:

$$\int_0^x \frac{\Delta\sigma_{zr}(r)}{\sqrt{x^2 - r^2}}\,dr = \frac{1}{\pi}\int_0^a \eta(s)\left[\frac{1}{s+x} - \frac{1}{s-x}\right]ds. \tag{63}$$

Furthermore, we introduce two new functions, denoted as $\varphi(x)$ and $\psi(x)$, as the even and odd extension of functions $\varphi(x)$ and $x\eta(x)$, respectively, namely,

$$\varphi(x) = \begin{cases} \varphi(x), & 0 \leqslant x < a, \\ \varphi(-x), & -a < x \leqslant 0, \end{cases} \tag{64}$$

$$\psi(x) = \begin{cases} x\eta(x), & 0 \leqslant x < a, \\ x\eta(-x), & -a < x \leqslant 0. \end{cases} \tag{65}$$

Thus, the first integral in the left-hand side of Eq. (54) can be represented in the form

$$\int_0^x \frac{r\Delta\sigma_{zz}(r)}{\sqrt{x^2 - r^2}}\,dr = -\frac{1}{\pi}\int_{-a}^a \frac{\varphi(s)}{s-x}\,ds, \quad -a < x < a. \tag{66}$$

Moreover,

$$\int_0^a \Delta\sigma_{zr}(r)\,dr = -\frac{2}{\pi}\int_0^a \left(\frac{d}{dr}\int_r^a \frac{x\eta(x)}{\sqrt{x^2 - r^2}}\,dx\right)dr = \frac{2}{\pi}\int_0^a \frac{\psi(x)}{x}\,dx \tag{67}$$

follows from Eq. (61) and

$$x\int_0^x \frac{\Delta\sigma_{zr}(r)}{\sqrt{x^2 - r^2}}\,dr = \frac{x}{\pi}\int_0^a \eta(s)\left[\frac{1}{s+x} - \frac{1}{s-x}\right]ds = -\frac{x}{\pi}\int_{-a}^a \frac{\psi(s)}{s(s-x)}\,ds \tag{68}$$

follows from Eqs. (63) and (65). Hence, the terms in the bracket at the left-hand side of Eq. (53) becomes

$$x\int_0^x \frac{\Delta\sigma_{zr}(r)}{\sqrt{x^2 - r^2}}\,dr - \int_0^a \Delta\sigma_{zr}(r)\,dr = -\frac{1}{\pi}\int_{-a}^a \frac{\psi(s)}{s-x}\,ds, \quad -a < x < a. \tag{69}$$

By virtue of Eqs. (58), (59) and (66) and (69), the system of Eqs. (53) and (54) for $\Delta\sigma_{zz}(r)$ and $\Delta\sigma_{zr}(r)$ are then transformed into the system of coupled singular integral equations for $\varphi(x)$ and $\psi(x)$:

$$\alpha\varphi(x) - \frac{\beta}{\pi}\int_{-a}^{a}\frac{\psi(s)}{s-x}\,\mathrm{d}s = b_z(x), \quad -a < x < a, \tag{70}$$

$$\frac{\beta}{\pi}\int_{-a}^{a}\frac{\varphi(s)}{s-x}\,\mathrm{d}s + \alpha\psi(x) = b_r(x), \quad -a < x < a, \tag{71}$$

where $b_z(x)$ and $b_r(x)$ for $0 \leqslant x < a$ are given by Eq. (57), together with Eq. (19), and $b_z(x)$ and $b_r(x)$ for $-a < x \leqslant 0$ are defined as $b_z(-x)$ and $-b_r(-x)$, respectively.

By introducing a complex function $\chi(x) = \varphi(x) + \mathrm{i}\psi(x)$, $\mathrm{i} = \sqrt{-1}$, the system of equations for $\varphi(x)$ and $\psi(x)$ can by rewritten as a single singular integral equation for $\chi(x)$

$$\alpha\chi(x) - \frac{\beta}{\pi \mathrm{i}}\int_{-a}^{a}\frac{\chi(s)}{s-x}\,\mathrm{d}s = b(x), \quad -a < x < a, \tag{72}$$

where $b(x) = b_z(x) + \mathrm{i}b_r(x)$. Eq. (72) is a better-studied standard singular integral equation with the Cauchy kernel, which can be solved analytically. Since α and β are constants, given by Eq. (55), using the technique, outlined in Muskhelishvili (1953) and Kanwal (1971), we, after some manipulations, obtain the solution of Eq. (72) as

$$\chi(x) = \frac{\mathrm{i}b_r}{\sqrt{\alpha^2-\beta^2}}(x + 2a\gamma_0\mathrm{i})\left(\frac{a-x}{a+x}\right)^{\mathrm{i}\gamma_0}, \quad -a < x < a, \tag{73}$$

or

$$\varphi(x) = \frac{b_r}{\sqrt{\alpha^2-\beta^2}}[x\sin(2\gamma_0\theta(x)) - 2a\gamma_0\cos(2\gamma_0\theta(x))], \tag{74}$$

$$\psi(x) = \frac{b_r}{\sqrt{\alpha^2-\beta^2}}[x\cos(2\gamma_0\theta(x)) + 2a\gamma_0\sin(2\gamma_0\theta(x))], \tag{75}$$

where

$$\gamma_0 = \frac{1}{2\pi}\ln\frac{\alpha+\beta}{\alpha-\beta} = \frac{1}{2\pi}\ln\frac{\kappa_1(\mu_2+\kappa_2\mu_1)}{\kappa_2(\mu_1+\kappa_1\mu_2)}, \tag{76}$$

$$\theta(x) = \frac{1}{2}\ln\left|\frac{a+x}{a-x}\right|. \tag{77}$$

5. Asymptotic stress field near the interface inclusion boundary

Since the expressions for $\varphi(x)$ and $\psi(x)$ have been determined in the proceeding section, all physical quantities of concern to us can be evaluated and given in explicit form. For many purposes, it is desirable to determine the elastic field at the interface, in particular the asymptotic stress field near the interface inclusion boundary, because it is crucial in studying certain features of material behavior. This is done in what follows. First, we give the stresses and displacements at the interface in closed form. Substituting Eqs. (47) and (48) into Eqs. (31) and (32) for $z = 0$ yields

$$\sigma_{zz(1)}(r, 0^+) = \frac{\mu_1}{2}\left(\gamma_2 + \frac{\gamma_1}{\kappa_1}\right)\int_0^a \Delta\sigma_{zz}(s)s\,\mathrm{d}s \int_0^\infty \xi J_0(r\xi)J_0(s\xi)\,\mathrm{d}\xi + \frac{\mu_1}{2}\left(\gamma_2 - \frac{\gamma_1}{\kappa_1}\right)\int_0^a \Delta\sigma_{zr}(s)s\,\mathrm{d}s \times \int_0^\infty \xi J_0(r\xi)J_1(s\xi)\,\mathrm{d}\xi, \tag{78}$$

$$\sigma_{zr(1)}(r, 0^+) = \frac{\mu_1}{2}\left(\gamma_2 - \frac{\gamma_1}{\kappa_1}\right)\int_0^a \Delta\sigma_{zz}(s)s\,\mathrm{d}s \int_0^\infty \xi J_1(r\xi)J_0(s\xi)\,\mathrm{d}\xi + \frac{\mu_1}{2}\left(\gamma_2 + \frac{\gamma_1}{\kappa_1}\right)\int_0^a \Delta\sigma_{zr}(s)s\,\mathrm{d}s \times \int_0^\infty \xi J_1(r\xi)J_1(s\xi)\,\mathrm{d}\xi, \tag{79}$$

respectively.

In view of Eqs. (A.8), (A.9), (A.12) and (A.14), we transform Eq. (78) to

$$\sigma_{zz(1)}(r, 0^+) = \frac{\mu_1}{2}\left(\gamma_2 + \frac{\gamma_1}{\kappa_1}\right)\Delta\sigma_{zz}(r) + \frac{\mu_1}{\pi}\left(\gamma_2 - \frac{\gamma_1}{\kappa_1}\right)\frac{\mathrm{d}}{r\,\mathrm{d}r}\int_0^a \Delta\sigma_{zr}(s)\,\mathrm{d}s \times \int_0^{\min(r,s)} \frac{x^2\,\mathrm{d}x}{\sqrt{r^2 - x^2}\sqrt{s^2 - x^2}}. \tag{80}$$

Reversing the order of the integration appearing in the last term and using Eq. (61), we find

$$\int_0^a \Delta\sigma_{zr}(s)\,\mathrm{d}s \int_0^{\min(r,s)} \frac{x^2\,\mathrm{d}x}{\sqrt{r^2 - x^2}\sqrt{s^2 - x^2}} = \int_0^r \frac{x^2\,\mathrm{d}x}{\sqrt{r^2 - x^2}} \int_x^a \frac{\Delta\sigma_{zr}(s)\,\mathrm{d}s}{\sqrt{s^2 - x^2}} = \int_0^r \frac{x\psi(x)\,\mathrm{d}x}{\sqrt{r^2 - x^2}}, \tag{81}$$

so by substituting Eqs. (60) and (81) into Eq. (80), the stress component $\sigma_{zz(1)}(r, 0^+)$ at the interface can be expressed in terms of $\varphi(x)$ and $\psi(x)$ as

$$\sigma_{zz(1)}(r, 0^+) = -\frac{\mu_1}{\pi}\frac{\mathrm{d}}{r\,\mathrm{d}r}\left[\left(\gamma_2 + \frac{\gamma_1}{\kappa_1}\right)\int_r^a \frac{x\varphi(x)}{\sqrt{x^2 - r^2}}\,\mathrm{d}x - \left(\gamma_2 - \frac{\gamma_1}{\kappa_1}\right)\int_0^r \frac{x\psi(x)}{\sqrt{r^2 - x^2}}\,\mathrm{d}x\right], \quad r \geqslant 0. \tag{82}$$

Using results (A.13) and (A.10), likewise from Eq. (79) the stress component $\sigma_{zr(1)}(r, 0^+)$ at the interface is given by

$$\sigma_{zr(1)}(r, 0^+) = -\frac{\mu_1}{\pi}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\gamma_2 + \frac{\gamma_1}{\kappa_1}\right)\int_r^a \frac{\psi(x)}{\sqrt{x^2 - r^2}}\,\mathrm{d}x + \left(\gamma_2 - \frac{\gamma_1}{\kappa_1}\right)\int_0^r \frac{\varphi(x)}{\sqrt{r^2 - x^2}}\,\mathrm{d}x\right], \quad r \geqslant 0. \tag{83}$$

Proceeding as before, we can deduce that the stress component $\sigma_{zz(2)}(r, 0^-)$ and $\sigma_{zr(2)}(r, 0^-)$ at the interface. They are

$$\sigma_{zz(2)}(r, 0^-) = \frac{\mu_2}{\pi}\frac{\mathrm{d}}{r\,\mathrm{d}r}\left[\left(\gamma_1 + \frac{\gamma_2}{\kappa_2}\right)\int_r^a \frac{x\varphi(x)}{\sqrt{x^2 - r^2}}\,\mathrm{d}x + \left(\gamma_1 - \frac{\gamma_2}{\kappa_2}\right)\int_0^r \frac{x\psi(x)}{\sqrt{r^2 - x^2}}\,\mathrm{d}x\right], \quad r \geqslant 0, \tag{84}$$

$$\sigma_{zr(2)}(r, 0^-) = \frac{\mu_2}{\pi}\frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\gamma_1 + \frac{\gamma_2}{\kappa_2}\right)\int_r^a \frac{\psi(x)}{\sqrt{x^2 - r^2}}\,\mathrm{d}x - \left(\gamma_1 - \frac{\gamma_2}{\kappa_2}\right)\int_0^r \frac{\varphi(x)}{\sqrt{r^2 - x^2}}\,\mathrm{d}x\right], \quad r \geqslant 0. \tag{85}$$

Moreover, an entirely similar procedure, considering Eqs. (A.9), (A.10) and (A.11), allows us to deduce the displacements at the interface in terms of $\varphi(x)$ and $\psi(x)$ as follows:

$$u_z(r, 0) = -\frac{\gamma_1 + \gamma_2}{2\pi}\int_0^r \frac{\varphi(x)}{\sqrt{r^2 - x^2}}\,\mathrm{d}x + \frac{\gamma_1 - \gamma_2}{2\pi}\int_r^a \frac{\psi(x)}{\sqrt{x^2 - r^2}}\,\mathrm{d}x, \quad r \geqslant 0, \tag{86}$$

$$u_r(r,0) = -\frac{\gamma_1-\gamma_2}{2\pi}\frac{1}{r}\int_r^a \frac{x\varphi(x)}{\sqrt{x^2-r^2}}\,\mathrm{d}x - \frac{\gamma_1+\gamma_2}{2\pi}\frac{1}{r}\int_0^r \frac{x\psi(x)}{\sqrt{r^2-x^2}}\,\mathrm{d}x, \quad r \geqslant 0. \tag{87}$$

In general, the analytical expressions for the stresses and displacements at the interface can be calculated through Eqs. (82)–(87) by using the method given by Spence (1968). However, these expressions are quite complicated and need not be written in explicit form. Next, we focus on getting the asymptotic stress field near the interface inclusion boundary.

On the one hand, owing to

$$\mu_1\left(\gamma_2 - \frac{\gamma_1}{\kappa_1}\right) = \mu_2\left(\gamma_1 - \frac{\gamma_2}{\kappa_2}\right), \tag{88}$$

by subtracting Eqs. (84) and (85) from Eqs. (82) and (83), respectively, we find that Eqs. (60) and (61) will be recovered respectively and it reveals the validity of the derivation of the above relations. Moreover, not only the asymptotic field for the stress jumps $\Delta\sigma_{zz}(r)$ and $\Delta\sigma_{zr}(r)$ across the inclusion can be calculated through Eqs. (60) and (61), but also the stresses $\sigma_{zz(1)}(r,0^+), \sigma_{zr(1)}(r,0^+)$ and $\sigma_{zz(2)}(r,0^-), \sigma_{zr(2)}(r,0^-)$ at the interface inclusion plane can be evaluated through Eqs. (82)–(85). In effect, in view of Eqs. (C.1) and (C.2), we can rewrite the stress $\sigma_{zz(1)}(r,0^+)$ along the inclusion as derivatives of related integrals from 0 to r, namely

$$\sigma_{zz(1)}(r,0^+) = C + \frac{\mu_1}{\pi}\left[-\coth(\pi\gamma_0)\left(\gamma_2+\frac{\gamma_1}{\kappa_1}\right) + \left(\gamma_2-\frac{\gamma_1}{\kappa_1}\right)\right]\frac{\mathrm{d}}{r\,\mathrm{d}r}\int_0^r \frac{x\psi(x)}{\sqrt{r^2-x^2}}\,\mathrm{d}x, \quad r < a, \tag{89}$$

where C is a constant, which can be uniquely determined and need not be written in explicit form since it does not disturb the distribution of stress.

By use of integration by parts and then differentiation, we find

$$\frac{\mathrm{d}}{r\,\mathrm{d}r}\int_0^r \frac{x\psi(x)}{\sqrt{r^2-x^2}}\,\mathrm{d}x = \int_0^r \frac{\psi'(x)}{\sqrt{r^2-x^2}}\,\mathrm{d}x. \tag{90}$$

It is seen from Eqs. (74) and (75) that a straightforward evaluation of $\psi'(x)$ yields

$$\psi'(x) = \frac{b_r\cos(2\gamma_0\theta(x))}{\sqrt{\alpha^2-\beta^2}} + \gamma_0\varphi(x)\left(\frac{1}{x-a} - \frac{1}{x+a}\right), \tag{91}$$

from which and Eq. (90), we get that the asymptotic stress $\sigma_{zz(1)}(r,0^+)$ at the upper surface of an inclusion can be expressed as

$$\sigma_{zz(1)}(r,0^+) = -\frac{\mu_1\gamma_0}{\pi}\left[\coth(\pi\gamma_0)\left(\gamma_2+\frac{\gamma_1}{\kappa_1}\right) - \left(\gamma_2-\frac{\gamma_1}{\kappa_1}\right)\right]\int_{-r}^r \frac{\varphi(x)}{\sqrt{r^2-x^2}(x-a)}\,\mathrm{d}x + \mathrm{O}(1), \quad r \simeq a^-, \tag{92}$$

where we have neglected some lower-order terms, which does not give rise to the singularity of stress. Substituting Eq. (74) into Eq. (92) and making use of the known results (Muskhelishvili, 1953):

$$\int_{-r}^r \frac{\cos(2\gamma_0\theta(x))}{\sqrt{r^2-x^2}(x-a)}\,\mathrm{d}x = -\frac{\pi\sqrt{\alpha^2-\beta^2}}{\alpha}\frac{\cos(2\gamma_0\theta(r))}{\sqrt{a^2-r^2}}, \quad r < a, \tag{93}$$

$$\int_{-r}^r \frac{\sin(2\gamma_0\theta(x))}{\sqrt{r^2-x^2}(x-a)}\,\mathrm{d}x = -\frac{\pi\sqrt{\alpha^2-\beta^2}}{\alpha}\frac{\sin(2\gamma_0\theta(r))}{\sqrt{a^2-r^2}}, \quad r < a \tag{94}$$

yield the asymptotic behavior of $\sigma_{zz(1)}(r,0^+)$ at the upper surface of an inclusion as follows:

$$\sigma_{zz(1)}(r,0^+) = ab_r\gamma_0\mu_1\left[\frac{1}{\beta}\left(\gamma_2+\frac{\gamma_1}{\kappa_1}\right)-\frac{1}{\alpha}\left(\gamma_2-\frac{\gamma_1}{\kappa_1}\right)\right]\frac{\sin(2\gamma_0\theta(r))-2\gamma_0\cos(2\gamma_0\theta(r))}{\sqrt{a^2-r^2}}+\mathrm{O}(1). \tag{95}$$

Performing as the above, the asymptotic expression for $\sigma_{zr(1)}(r,0^+)$ near the inclusion front along the inclusion is given by

$$\sigma_{zr(1)}(r,0^+) = ab_r\gamma_0\mu_1\left[\frac{1}{\beta}\left(\gamma_2+\frac{\gamma_1}{\kappa_1}\right)-\frac{1}{\alpha}\left(\gamma_2-\frac{\gamma_1}{\kappa_1}\right)\right]\frac{\cos(2\gamma_0\theta(r))+2\gamma_0\sin(2\gamma_0\theta(r))}{\sqrt{a^2-r^2}}+\mathrm{O}(1). \tag{96}$$

If we denote $\rho = a - r$, ρ being the distance from the inclusion front, we have that as $\rho \ll 1$,

$$\sigma_{zz(1)}(\rho,0^+)+\mathrm{i}\sigma_{zr(1)}(\rho,0^+) = \frac{\mathrm{i}b_r\gamma_0\mu_1}{2}\left[\frac{1}{\beta}\left(\gamma_2+\frac{\gamma_1}{\kappa_1}\right)-\frac{1}{\alpha}\left(\gamma_2-\frac{\gamma_1}{\kappa_1}\right)\right](1+2\gamma_0\mathrm{i})\rho^{-(1/2)+\mathrm{i}\gamma_0}(2a)^{(1/2)-\mathrm{i}\gamma_0}+\mathrm{O}(1). \tag{97}$$

In a similar manner, we can derive the asymptotic stress field

$$\sigma_{zz(2)}(\rho,0^-)+\mathrm{i}\sigma_{zr(2)}(\rho,0^-) = -\frac{\mathrm{i}b_r\gamma_0\mu_2}{2}\left[\frac{1}{\beta}\left(\gamma_1+\frac{\gamma_2}{\kappa_2}\right)+\frac{1}{\alpha}\left(\gamma_1-\frac{\gamma_2}{\kappa_2}\right)\right] \times (1+2\gamma_0\mathrm{i})\rho^{-(1/2)+\mathrm{i}\gamma_0}(2a)^{(1/2)-\mathrm{i}\gamma_0}+\mathrm{O}(1). \tag{98}$$

Obviously, the normal and shear stresses near the inclusion front exhibit the oscillatory singularity, which is divergent in both amplitude and frequency as $r \simeq a^-$ or $\rho \ll 1$. Moreover, subtracting Eq. (98) from Eq. (97) results in the asymptotic expressions for the stress jumps $\Delta\sigma_{zz}(r)$ and $\Delta\sigma_{zr}(r)$ near the inclusion front, given by

$$\Delta\sigma_{zz}(\rho)+\mathrm{i}\Delta\sigma_{zr}(\rho) = \frac{\mathrm{i}b_r\gamma_0}{\beta}(1+2\gamma_0\mathrm{i})\rho^{-(1/2)+\mathrm{i}\gamma_0}(2a)^{(1/2)-\mathrm{i}\gamma_0}+\mathrm{O}(1), \tag{99}$$

from which it is observed that the normal and shear stress jumps possess the same oscillatory singularity. Here we have utilized Eq. (88) and the following equality

$$\mu_1\left(\gamma_2+\frac{\gamma_1}{\kappa_1}\right)+\mu_2\left(\gamma_1+\frac{\gamma_2}{\kappa_2}\right)=2. \tag{100}$$

6. Discussion and conclusion

For a rigid circular inclusion at the interface of a bimaterial, it is readily seen that, apart from the inverse square root singularity, the normal and shear stresses near the inclusion front at the inclusion surfaces possess simultaneously the oscillatory singularity. It reveals that, to order to fulfill the ideal mixed boundary conditions, there exist simultaneously the pressure and traction exerted at the inclusion surfaces, which alternating change and exhibit the oscillatory feature. Also, when the sign of normal stress varies, the sign of shear stress on the inclusion surface changes simultaneously and the characteristic of shear stress is similar to that of normal stress. Furthermore, from Eq. (99) the magnitude of the singularity of the stress jumps depends on material constants of the upper and lower half spaces.

In comparison with three-dimensional interface crack problems, we see that for an interfacial circular crack of a bimaterial, the singularity depends only on the axial tensions at infinity or the internal pressure at the crack surfaces (Erdogan, 1965), whereas it for an interfacial rigid inclusion depends on the axial and radial tensions at infinity. In other words, remote tensile loads parallel to and perpendicular to the inclusion surface have effect on the distribution of stress field for interface inclusion problems, while remote tensile load parallel to the crack surface has no effect on the stress distribution and remote tensile load perpendicular to the crack surface gives rise to the singularity of stresses near the crack front for interface crack problems. And the bimaterial elastic constant or the oscillation factor for rigid interface inclusion problems is

$$\gamma_0 = \frac{1}{2\pi} \ln \frac{\alpha + \beta}{\alpha - \beta} = \frac{1}{2\pi} \ln \frac{\kappa_1(\mu_2 + \kappa_2\mu_1)}{\kappa_2(\mu_1 + \kappa_1\mu_2)}, \tag{101}$$

while for interface crack problems (Kassir and Bregman, 1972) it is

$$\gamma_0 = \frac{1}{2\pi} \ln \frac{\mu_2 + \kappa_2\mu_1}{\mu_1 + \kappa_1\mu_2}. \tag{102}$$

Moreover, a comparison of the bimaterial elastic constant for three-dimensional interfacial rigid inclusion with its counterpart for two-dimensional case (see e.g. Ballarini, 1990; Jiang and Liu, 1992; etc.) reveals that the dependence of bimaterial elastic constant on elastic constants of two dissimilar materials is the same. However, it is noted that γ_0 in some papers is replaced with $-\gamma_0$, and that it may be expressed in terms of two Dundurs's constants. For a particular case of bimaterial consisting of steel occupying the upper half space and glass occupying the lower half space, the relevant elastic constants are $E_1 = 3 \times 10^7$ psi, $v_1 = 0.3$, and $E_2 = 10^7$ psi, $v_2 = 0.22$, so we can evaluate the bimaterial elastic constant $\gamma_0 = 0.03953$ for interfacial rigid inclusion problems through Eq. (101) and $\gamma_0 = 0.06557$ (Kassir and Bregman, 1972) interfacial crack problems through Eq. (102).

In what follows, we consider two special cases: one corresponding to that the upper and lower half spaces are of the same material and the other involving the problem of a rigid circular plate bonded adhesively to the surface of an elastic half space subjected to pure radial tension at infinity. For the former, we have $\mu_1 = \mu_2$, $v_1 = v_2$, and further $\gamma_1 = \gamma_2$, $\varphi(x) = 0$, $\psi(x) = b_r x/\alpha$. Thus putting these values into the expressions for the asymptotic stress field, we see that they reduce to indeterminacies. In order to overcome this difficult, we let $\gamma_0 \to 0$ and $\beta\ (= \gamma_1 - \gamma_2) \to 0$ in the asymptotic field and obtain

$$\Delta\sigma_{zz}(\rho) + \mathrm{i}\Delta\sigma_{zr}(\rho) = \frac{\mathrm{i}b_r}{\pi\alpha}\rho^{-(1/2)}(2a)^{1/2} + \mathrm{O}(1) = \frac{4\mathrm{i}(1-v)}{\pi(3-4v)(1+v)}[(1-v)\sigma_{rr}^{\infty} + v\sigma_{zz}^{\infty}]\rho^{-(1/2)}(2a)^{1/2} + \mathrm{O}(1) \tag{103}$$

in accordance with that determined directly from Eqs. (60) and (61).

It indicates that for a rigid circular inclusion embedded in an infinite elastic solid subjected to constant tension along the axial and radial directions at infinity, the normal stress is continuous across a rigid inclusion and the radial shear stress jump across an inclusion near the inclusion front exhibits the inverse square-root singularity. Under such circumstances, one would expect that the radial shear stress near the inclusion front is so high that it may result in the material cracking.

Another special case to be considered is that a rigid circular flat plate is bonded to the surface of a semi-infinite elastic material, denoted as the upper half space, subjected to remote tension along the radial direction at infinity. For this case, it suffices to take the value of μ_2 of the lower half-space material as zero in the above-derived solution. Hence, we obtain the distribution of stress near the inclusion front below,

$$\sigma_{zz}(\rho) + \mathrm{i}\sigma_{zr}(\rho) = \frac{\mathrm{i}\sigma_{rr}^{\infty}(1-v_1)\ln\kappa_1}{2\pi(1+v_1)(1-2v_1)}\left(1 + \frac{\mathrm{i}\ln\kappa_1}{\pi}\right)\rho^{-(1/2)+\mathrm{i}(\ln\kappa_1/\pi)}(2a)^{(1/2)-\mathrm{i}(\ln\kappa_1/\pi)} + \mathrm{O}(1), \quad \rho = a - r \ll 1. \tag{104}$$

Finally, we summarize the main conclusions obtained in this paper as follows:

(1) The asymptotic stress field near the interfacial rigid circular inclusion front of a bimaterial exhibits the oscillatory singularity, apart from the inverse square-root singularity, similar to the characteristics of stress field for an interfacial circular crack. However, the dependence relationship of the bimaterial elastic

constant on elastic constants of two bonded dissimilar materials for interfacial rigid inclusion is different from that for interfacial crack.

(2) The singular field of tangent and normal stresses can be induced by remote tensile loads parallel to or (and) perpendicular to the inclusion surface for interfacial rigid inclusion problems, while it can be caused only by remote tensile loads perpendicular to the crack surface for interfacial crack problems.

(3) For a rigid circular inclusion embedded in an infinite elastic medium, both the axial and the radial tensions at infinity give rise to the shear stress of the inverse square-root singularity near the inclusion front at the inclusion surfaces, while for a circular crack contained in an infinite elastic medium, only the axial tensions at infinity can produce the normal stress of the inverse square-root singularity near the crack front.

(4) For a rigid circular plate bonded to the surface of a semi-infinite elastic material subjected to remote tension along the radial direction at infinity, in addition to the inverse square-root singularity, stress field underneath a rigid flat plate possesses the nature of the oscillatory singularity.

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Appendix A

Some useful results in this paper are listed below, which can be found in Gradshteyn and Ryzhik (1980),

$$\int_0^\infty J_0(r\xi)\cos(x\xi)\,\mathrm{d}\xi = \begin{cases} \frac{1}{\sqrt{r^2-x^2}}, & r > x, \\ 0, & r < x, \end{cases} \tag{A.1}$$

$$\int_0^\infty J_0(r\xi)\sin(x\xi)\,\mathrm{d}\xi = \begin{cases} 0, & r > x, \\ \frac{1}{\sqrt{x^2-r^2}}, & r < x, \end{cases} \tag{A.2}$$

$$\int_0^\infty J_1(r\xi)\cos(x\xi)\,\mathrm{d}\xi = \begin{cases} \frac{1}{r}, & r > x, \\ \frac{1}{r} - \frac{x}{r\sqrt{x^2-r^2}}, & r < x, \end{cases} \tag{A.3}$$

$$\int_0^\infty J_1(r\xi)\sin(x\xi)\,\mathrm{d}\xi = \begin{cases} \frac{x}{r\sqrt{r^2-x^2}}, & r > x, \\ 0, & r < x, \end{cases} \tag{A.4}$$

$$\int_0^x \frac{rJ_0(r\xi)}{\sqrt{x^2-r^2}}\,\mathrm{d}r = \frac{\sin(x\xi)}{\xi}, \tag{A.5}$$

$$\int_\rho^r \frac{x}{\sqrt{x^2-\rho^2}\sqrt{r^2-x^2}}\,\mathrm{d}x = \frac{\pi}{2}, \tag{A.6}$$

$$\int_0^{\min(x,s)} \frac{r}{\sqrt{x^2-r^2}\sqrt{s^2-r^2}}\,\mathrm{d}r = \frac{1}{2}\ln\left|\frac{s+x}{s-x}\right|, \tag{A.7}$$

$$\rho\int_0^\infty \xi J_0(r\xi)J_0(\rho\xi)\,\mathrm{d}\xi = \delta(\rho-r), \quad \delta(x) \text{ being the Dirac delta function} \tag{A.8}$$

$$\int_0^\infty \xi J_0(r\xi)J_1(\rho\xi)\,\mathrm{d}\xi = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\int_0^\infty J_1(r\xi)J_1(\rho\xi)\,\mathrm{d}\xi\right], \tag{A.9}$$

$$\int_0^\infty J_0(r\xi)J_0(\rho\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\int_0^{\min(r,\rho)} \frac{\mathrm{d}s}{\sqrt{r^2-s^2}\sqrt{\rho^2-s^2}}, \tag{A.10}$$

$$\int_0^\infty J_0(r\xi)J_1(\rho\xi)\,\mathrm{d}\xi = \begin{cases} 0, & \rho < r, \\ \frac{1}{\rho}, & \rho > r, \end{cases} \tag{A.11}$$

$$\int_0^\infty J_1(r\xi)J_1(\rho\xi)\,\mathrm{d}\xi = \frac{2}{\pi}\frac{1}{r\rho}\int_0^{\min(r,\rho)} \frac{s^2\,\mathrm{d}s}{\sqrt{r^2-s^2}\sqrt{\rho^2-s^2}}, \tag{A.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}J_0(r\xi) = -\xi J_1(r\xi), \tag{A.13}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}[rJ_1(r\xi)] = r\xi J_0(r\xi). \tag{A.14}$$

Appendix B

To derive Eq. (62) we perform the following transformation

$$\begin{aligned}
\int_0^x \frac{r\,\Delta\sigma_{zz}(r)}{\sqrt{x^2-r^2}}\,\mathrm{d}r &= \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x r\,\Delta\sigma_{zz}(r)\sqrt{x^2-r^2}\,\mathrm{d}r \\
&= -\frac{2}{\pi}\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \sqrt{x^2-r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\int_r^a \frac{s\varphi(s)}{\sqrt{s^2-r^2}}\,\mathrm{d}s\right)\mathrm{d}r \\
&= -\frac{2}{\pi}\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left[-x\int_0^a \varphi(s)\,\mathrm{d}s + \int_0^a s\varphi(s)\,\mathrm{d}s\int_0^{\min(x,s)} \frac{r}{\sqrt{x^2-r^2}\sqrt{s^2-r^2}}\,\mathrm{d}r\right] \\
&= \frac{2}{\pi}\frac{1}{x}\left[\int_0^a \varphi(s)\,\mathrm{d}s + \frac{1}{2}\int_0^a s\varphi(s)\left(\frac{1}{s+x}-\frac{1}{s-x}\right)\mathrm{d}s\right] \\
&= \frac{1}{\pi}\int_0^a \varphi(s)\left[\frac{1}{s+x}-\frac{1}{s-x}\right]\mathrm{d}s,
\end{aligned}$$

where at the third step integration by parts has been used, and at the fourth step the result (A.7) has been taken into account.

Appendix C

In order to evaluate the asymptotic stress field, it is convenient to exploit the following relationships, given in Spence (1968):

$$\frac{2}{\pi}\times\cosh(\pi\gamma_0)\int_0^r \frac{f(x)\,\cos(2\gamma_0\theta(x))}{\sqrt{r^2-x^2}}\,\mathrm{d}x - \frac{2}{\pi}\sinh(\pi\gamma_0)\int_r^a \frac{f(x)\,\sin(2\gamma_0\theta(x))}{\sqrt{x^2-r^2}}\,\mathrm{d}x = L_0[f(x)], \tag{C.1}$$

$$\frac{2}{\pi}\cosh(\pi\gamma_0)\int_0^r \frac{xf(x)\,\sin(2\gamma_0\theta(x))}{\sqrt{r^2-x^2}}\,dx + \frac{2}{\pi}\sinh(\pi\gamma_0)\int_r^a \frac{xf(x)\,\cos(2\gamma_0\theta(x))}{\sqrt{x^2-r^2}}\,dx = L_1[f(x)], \tag{C.2}$$

where $f(x)$ is an even function and has no singularity over $[-a, a]$. Moreover, a direct evaluation of $L_0[f(x)]$ and $L_1[f(x)]$ by applying the residue theorem, as given in Spence (1968), for some particular functions gives some discrete results below:

$$L_0[1] = 1, \qquad L_0[x^2] = \frac{r^2}{2} - 2\gamma_0^2, \tag{C.3}$$

$$L_1[1] = 2\gamma_0, \qquad L_1[x^2] = \gamma_0 r^2 - \tfrac{2}{3}\gamma_0(5 + 2\gamma_0^2) \tag{C.4}$$

and so forth. It is noted that the notation here is slightly different from that used in Spence (1968).

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